MATH 54 – MIDTERM 2 – BONUS SOLUTIONS

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Bonus: In this problem, we're going to use the Wronskian to find the general solution of a quite complicated differential equation! This should illustrate yet again the power of the Wronskian!

(a) Consider the differential equation:

$$y'' + P(t)y' + Q(t)y = 0$$

Recall the definition of the Wronskian determinant:

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_2'(t)y_1(t) - y_1'(t)y_2(t)$$

Where y_1 and y_2 solve the above differential equation.

By differentiating $W(t)$ with respect to t, find a simple differential equation satisfied by $W(t)$ and solve it. You answer will involve the $\int$ sign!

$$\begin{aligned} W'(t) =& y_2''(t)y_1(t) + \cancel{y_2'(t)y_1'(t)} - y_1''(t)y_2(t) - \cancel{y_1'(t)y_2'(t)} \\ =& y_2''(t)y_1(t) - y_1''(t)y_2(t) \\ =& \left(-P(t)y_2'(t) - Q(t)y_2(t)\right) y_1(t) + \left(P(t)y_1'(t) + Q(t)y_1(t)\right) y_2(t) \\ =& - P(t)y_2'(t)y_1(t) - \cancel{Q(t)y_1(t)y_2(t)} + P(t)y_1'(t)y_2(t) + \cancel{Q(t)y_1(t)y_2(t)} \\ =& - P(t)\left(y_2'(t)y_1(t) - y_1'(t)y_2(t)\right) \\ =& - P(t)W(t) \end{aligned}$$

Hence $W'(t) = -P(t)W(t)$, so $\boxed{W(t) = Ce^{-\int P(t)dt}}$

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(b) Hence, by (a), we know:

$$y_2'(t)y_1(t) - y_2(t)y_1'(t) = \text{your answer in } (a)$$

Divide this equality by $(y_1(t))^2$ and recognize the left-hand-side as the derivative of a quotient, and hence solve for y_2 in terms of y_1. You answer will involve another $\int$ sign!

We have:

$$\begin{aligned} y_2'(t)y_1(t) - y_2(t)y_1'(t) &= e^{-\int P(t)dt} \\ \frac{y_2'(t)y_1(t) - y_2(t)y_1'(t)}{(y_1(t))^2} &= \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\ \left(\frac{y_2(t)}{y_1(t)}\right)' &= \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\ \frac{y_2(t)}{y_1(t)} &= \int \frac{e^{-\int P(t)dt}}{(y_1(t))^2} \\ y_2(t) &= \left(\int \frac{e^{-\int P(t)dt}}{(y_1(t))^2}\right) y_1(t) \end{aligned}$$

(c) Let's apply the result in (b) to the differential equation:

$$y'' - \tan(t)y' + 2y = 0$$

(here $P(t) = -\tan(t)$, $Q(t) = 2$)

One solution (by guessing) is given by $y_1(t) = \sin(t)$. Use your answer to (b) to find *another* solution $y_2(t)$!

Hint: You may use the following facts: $\int \tan(t)dt = -\ln(\cos(t))$, the substitution $u = \frac{1}{\sin(t)}$, and finally the formula $\frac{u^2}{1-u^2} = \frac{1}{1-u^2} - 1 = \frac{1}{2(1-u)} + \frac{1}{2(1+u)} - 1$.

We have:

$$
\begin{aligned}
y_2(t) &= \left(\int \frac{e^{-\int P(t)dt}}{(y_1(t))^2}\right) y_1(t) \\
&= \left(\int \frac{e^{\int \tan(t)dt}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{e^{-\ln(\cos(t))}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{\frac{1}{e^{\ln(\cos(t))}}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{\frac{1}{\cos(t)}}{\sin^2(t)}\right) \sin(t) \\
&= \left(\int \frac{1}{\cos(t)\sin^2(t)}\right) \sin(t) \\
&= \left(\int \left(\frac{1}{\cos^2(t)}\right)\left(\frac{\cos(t)}{\sin^2(t)}\right)\right) \sin(t) \\
&= \left(\int \left(\frac{1}{1-\sin^2(t)}\right)\left(\frac{\cos(t)}{\sin^2(t)}\right)\right) \sin(t) \\
&= \left(\int \frac{-du}{1-\frac{1}{u^2}}\right) \sin(t) \qquad \text{Use } u = \frac{1}{\sin(x)}\text{, then } \sin(x) = \frac{1}{u} \\
&= \left(\int \frac{-u^2}{u^2-1} du\right) \sin(t) \qquad \text{(multiply top and bottom by } u^2\text{)} \\
&= \left(\int -1 + \frac{-1}{2(u-1)} + \frac{1}{2(1+u)}\right) \sin(t) \\
&= \left(-u + \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1|\right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left|\frac{1}{\sin(t)}+1\right| - \frac{1}{2}\ln\left|\frac{1}{\sin(t)}-1\right|\right) \sin(t) \\
&= \left(\frac{-1}{\sin(t)} + \frac{1}{2}\ln\left|\frac{\frac{1}{\sin(t)}+1}{\frac{1}{\sin(t)}-1}\right|\right) \sin(t) \\
&= \left(\frac{1}{-\sin(t)} + \frac{1}{2}\ln\left|\frac{1+\sin(t)}{1-\sin(t)}\right|\right) \sin(t) \qquad \text{(multiply top and bottom by } \sin(t)\text{)} \\
&= -1 + \sin(t)\coth^{-1}(\sin(t))
\end{aligned}
$$

(d) Notice that the equation $y'' - \tan(t)y' + 2y = 0$, although quite complicated, is still *linear*. What is the general solution of $y'' - \tan(t)y' + 2y = 0$?

$$y(t) = Ay_1(t) + By_2(t) = A\sin(t) + B(-1 + \sin(t)\coth^{-1}(\sin(t)))$$